The analysis of errors in the solution of ordinary differential equations

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Abstract— **In this paper we present the results of an investigation carried out with students of 2nd Technical Engineering in Computer Management, which studied whether the continued use of a methodology based on error analysis improves their academic performance in the study of ordinary differential equations. The sample consisted of a total of 18 students in a Spanish public university. The investigation design was quasi-experimental comparison with an experimental group and a control group. The results allow to assert that if it is continuously used the aforementioned methodology based on the analysis of student's errors, it increases academic performance of students when they study ordinary differential equations in a more effective way than if it follows the traditional methodology.**

Keywords-ordinary differential equations; analysis of errors; mathematics; pedagogy.

I. Introduction

Ordinary differential equations are a basic tool for professionals in fields related to science and technology such as computer engineers. This is due to ordinary differential equations allow describing phenomena based on variation and therefore allow us modeling and solving problems from a variety of contexts.

Reference [1] conducted a study in order to explore the teaching and learning of first-order ordinary differential equations through a qualitative study of their solutions. Reference [9] identified the different strategies used by students to solve ordinary differential equations, and [14] developed a framework for interpreting and understanding the difficulties that students have with central mathematical ideas in differential equations. Reference [5] found that, in general, the idea that students have to solve a differential equation is reduced to the application of specific algorithms for classification and solving ordinary differential equations. Reference [18] conducted a review of various works related to teaching and learning of ordinary differential equations, identifying different strategies, difficulties and understandings that students show related to the creation, interpretation and coordination of different systems of representation (including phase and bifurcation diagrams) and formulating justified predictions about the behavior of the solutions.

A teaching model proposed in the field of investigation in mathematics education is conducted in the framework of the Inquiry-Oriented Differential Equation (IO-DE), in which the concept of ordinary differential equation is interpreted as an expression that shows the evolution of time function, as in [16]. The students who participated in the project worked in an environment that promoted discussion, the approach of conjecture, justification of ideas and creating own resolution methods, as in [15]. Reference [17] states that several investigations have shown that these students perform better results than others who did not participate in this project, especially in activities related to modeling and analysis of the behavior of solutions of an equation, as well as having a greater capacity to retain knowledge and math skills, as described in [11].

It is a fact that our students make mistakes in their productions when working any mathematical domain, in particular, ordinary differential equations. The analysis of those mistakes is a valuable source of information considered as Weiner (quoted by [13]), who started a educational investigation oriented to the study of errors. There are numerous investigations where errors are classified and categorized using different approaches ([6], [4], [19], [7], [3], [2]). Reference [20] considers three axes, not disjoint, that allow us to analyze the source of the error. In this way, we can classify student's errors on three different origins: obstacle (no use of parentheses, concatenation, need for closure), meaninglessness (arithmetic: no use of parentheses, particularization; procedure: improper distributive property) and affective and emotional attitudes. Reference [10] considers that among these difficulties that exist in the teaching-learning process of differential equations, it has been found obstacles in the integration of different registers of representation.

We consider that it is interesting to propose tools to improve the teaching and learning of ordinary differential equations in the first college courses careers in science and engineering, which is an issue raised by [8]. One such tool is the analysis of errors. When discussing why something is wrong, the student must maintain focus on the arguments made by their partners and bring their own, and that is why it is necessary to think for himself. Error analysis is one of the most powerful tools. However, despite its potential, it is an instrument that is not used consistently in the classroom when teaching ordinary differential equations. This suggests an opportunity to extend their use, adapting in an adequate way to various situations, and levels of education courses as proposed in [12]. It is important to explore innovative strategies to develop performance in teaching-learning processes of ordinary differential equations in our students. It is also important to explore innovative strategies to develop the academic performance of our students in the learning processes of ordinary differential equations. Our investigation complements previous works on teaching and learning of ordinary differential equations as [8], and on the errors and difficulties that students make when they solve an ordinary differential equation as in [14]. Our goal is to study the relation between the application of a methodology based on the analysis of errors in a classroom with students of 2nd Technical Engineering in Computer Management, and the improvement of their academic performance in the study of ordinary differential equations. We consider that the results of this study will provide suggestions to increase existing teaching techniques to improve the teaching and learning of ordinary differential equations.

II. METHODOLOGY

A. Design

The investigation was conducted in the field of Technical Engineering in Computer Management degree offered by the University of Granada on the campus of Ceuta. Participants were 18 students, aged between 19 and 23 years old, who attended the Numerical Analysis course. The investigation design was quasi-experimental comparison with an experimental group and a control group. The experimental group consisted of 9 students as well as the control group.

B. Hypothesis

The hypothesis of our study is stated as follows: If we use continuously a methodology based on error analysis with students of 2nd Technical Engineering in Computer Management, then they will improve their academic performance when solving ordinary differential equations.

C. Variables

In this section we describe the variables used in our study. We include two variables in total.

- Independent variable. The methodology based on error analysis.
- Dependent variable. The academic performance of students when solving ordinary differential equations. This is a quantitative variable which range between 0 and 10 points.

D. Method

The study can be divided into three phases:

- Pretest phase. This phase took place in early March 2010. The initial test to determine their level of mathematical knowledge was applied to all students in the sample, simultaneously and in the same classroom. This test can be found in appendix A. During the application of this 1 hour test, students had to respond to a total of 5 questions that dealt with solving equations, numerical sets and calculation of limits, derivatives and integrals.
- Intervention phase. This phase was carried out by applying the methodology based on the analysis of errors in the experimental group during 26 sessions of 1 hour each over a total of four months. The timetable established for these sessions coincided in both groups. The contents explained to both groups were: definition of ordinary differential equation, resolution of first-order ordinary differential equations (separate variables, homogeneous, Bernoulli, exact differential) and ordinary differential equations with initial conditions. Once contents were explained to the students belonging to the experimental group, it was presented several solved activities containing different types of errors: errors when operating algebraically, logic errors, technical errors, misinterpreted data, incorrect use of language, and students had to detect reasonably the error. Next we describe the stages followed in the search of errors:

  - Opening: An activity is posed to students. We ensure that students know exactly what they have to do.

  - Implementation in which the proposal was made. The implementation could be: a large group or class group from a common dialogue, students in pairs or individually.

  - Contrasting: Stage where ideas were contrasted through dialogue. If the previous phase was carried out individually, this phase was carried out in pairs. If the execution was carried out in pairs, this phase is performed in pairs of pairs. If the previous phase had involved the class group, the stage at which we are now comprised of the above.

  - Exhibition: Stage where the class group intervened with free participation of each student who wanted to express his ideas. Through dialogue in large group and teacher's questions, ideas were channeled and collected mathematical strategies were recognized as invalid.

  - Completion: Students wrote the conclusions drawn: conceptual, procedural, etc.

  While students in the experimental group were carrying out these activities, students in the control group solved exercises of ordinary differential equations following the traditional methodology and did not analyze errors.
- Posttest phase. This stage took place during the month of June 2010. Students were administered the same day, at the same time and in the same classroom, the test of ordinary differential equations that appears in appendix B. This test was composed of three exercises; in the first one four ordinary differential equations had to be identified and solved; in the second one, two

questions relative to the concept of solution of an ordinary differential equation were made, and in the third one, an ordinary differential equation with initial conditions was given and the student had to answer two questions relative to the model described by the ordinary differential equation. The time allowed for the resolution was 2 hours.

III. RESULTS

In order to check if the pretest-postest change of the variable named academic performance (dependent variable) differed in each of the groups, with respect to the use or not, of the methodology based in the analysis of errors (independent variable) we checked the normality of both samples and next we used the Student's t-test and the Levene´s test in order to make the statistical hypothesis test. We also studied whether changes were significant and whether these were due to the use of the new methodology, verifying their impact in the experimental group. Statistical significance (Sig. F) was analyzed. The statistical hypothesis test was based on a contrast of equality of means of two normal samples with unknown variances. In the pretest phase it was made a bilateral test considering the following hypotheses:

- Null hypothesis: There will be no statistically significant differences between the results obtained by the experimental group and the control group for the variable level of mathematical knowledge.
- Alternative hypothesis: There will be statistically significant differences between the results obtained by the experimental group and the control group for the variable level of mathematical knowledge.

The existence of statistically significant differences would reject null hypothesis and accept the alternative one. The results of the statistical study on the pretest phase, shown in Table I, allow us to deduce that there are no statistically significant differences ($p<0.05$) between the experimental and control groups in terms of their level of mathematical knowledge.

TABLE I.

	F	***Sig. (2-tailed)***
Level of mathematical knowledge	0.893	0.424

In the posttest phase it was made a bilateral test considering the following hypotheses:

- Null hypothesis: There will be no statistically significant differences between the results obtained by the experimental group and the control group, for the variable academic performance in ordinary differential equations.
- Alternative hypothesis: There will be statistically significant differences between the results obtained by the experimental group and the control group for the variable academic performance in ordinary differential equations.

TABLE II.

	F	***Sig. (2-tailed)***
Academic performance in ordinary differential equations	4.919	0.003

As we can see in Table II, the experimental group posttest changes were statistically significant ($p<0.05$) compared to the control group posttest in the test of ordinary differential equations. In view of the results, the null hypothesis is rejected.

All analyzes were performed with the Statistical Package for Social Sciences (SPSS) version 15.0.

IV. CONCLUSION

The investigation presented in this article intended to verify that if we use continuously a methodology based on error analysis with students of 2nd Technical Engineering in Computer Management, then they will improve their academic performance when solving ordinary differential equations, and we have obtained a positive response confirming the hypothesis of our study.

In most cases the misconceptions stem from previous courses and its corrections are not an easy task, as stated [12]. With this study we show that when the methodology based on the analysis of errors is the main element considered in teaching ordinary differential equations, students assimilate concepts in a more effective way and get better grades on tests.

We also confirm that students take an active part in the discovery of errors with this methodology, and students who were accustomed to a passive memorization presented greater difficulties.

Trying to supplement previous investigations as carried out by [4] and [3] we want to emphasize, by its frequency, we have observed certain errors that students make when solving an ordinary differential equation and are the following:

- Product (quotient) linearization error that consists of calculating the integral of a product (quotient) of functions as the product (quotient) of the integrals.
- Identification error that consists of that the student does not recognize the type of ordinary differential equation correctly and this makes that he makes a mistake in the solving process.

In the future it would be desirable to confirm these results statistically expanding the sample with students on other courses.

V. APPENDIX A (TEST OF MATHEMATICAL KNOWLEDGE)

1. a) Define the absolute value of a real number.

b) Solve the equation $|x-3|=2$

2. a) Consider the following numbers: $0.\overset{\frown}{6}$, 0.666, $\sqrt{5}$, which of them are rational numbers?

b) What is the relation between $0.\overset{\frown}{9}$ and 1?

3. Calculate the following limits:

a) $\lim_{x\to -2} x^3 - 2x^2 - x + 5$

b) $\lim_{x\to 0} \frac{x^3 + 5x}{6x}$

c) $\lim_{x\to +\infty} \frac{e^{x+1}}{\ln(x^2+1)}$

where ln denotes the natural logarithm.

4. Calculate the derivative of the following functions:

a) $f(x) = e^{x^3-4x^2+x-1}$

b) $f(x) = \ln(3x^2 - x + 2)$

c) $f(x) = \sin(\cos(x))$

5. Calculate:

a) $\int x\ln(x)dx$

b) $\int_{-1}^{1} \frac{1}{x^2} dx$

VI. APPENDIX B (TEST OF ORDINARY DIFFERENTIAL EQUATIONS)

1. Identify and solve the following ordinary differential equations:

a) $y' = e^x - \frac{3x}{x^2-1}$

b) $\left(3t^2x + x^3\right)x' + 2t^3 = 0$

c) $xy' + y = y^2 \log(x)$

d) $t\cos\left(t+x\right) + sin\left(t+x\right) + t\cos\left(t+x\right)x' = 0$

2. Answer reasonably the following questions:

a) Can the function $f(t) = t^2$, with $t \in \mathrm{R}$, be the solution of an homogeneous first-order ordinary differential equation? And can it be a solution of a nonhomogeneous first-order ordinary differential?

b) Can the functions $f(t) = e^t$ and $g(t) = e^{-t}$, with $t \in \mathrm{R}$, be solutions of the same homogeneous first-order ordinary differential equation? And can they be solutions of a nonhomogeneous first-order ordinary differential?

3. The population $P(t)$ of a suburb of a large city is modeled by:

$$\begin{cases} \frac{dP}{dt} = P\left(10^{-1} - 10^{-7}P\right) \\ P(0) = 5000 \end{cases}$$

where t is measured in months.

a) What is the value limit of the population?
b) What moment will the population be equal to the half of its limit?

REFERENCES

[1] M. Artigue, “Functions from an Algebraic and Graphic Point of View: Cognitive Difficulties and Teaching Practices”, in Aspects of epistemology and Pedagogy, MAA notes 25, E. Dubinsky and G. Harel, Eds. Washington: MAA, pp. 109-132, 1992.

[2] J.P. Astolfi, El “error”, un medio para enseñar, Sevilla: Díada Editora, 1999.

[3] C. Azcárate, D. Bosch, M. Casadevall and E. Casellas, Cálculo diferencial e integral, España: Editorial Síntesis, 1996.

[4] L.R. Booth, Algebra: children’s strategies and errors, Windsor: NFER-Nelson, 1984.

[5] M. Camacho, J. Perdomo and M. Santos Trigo, “La resolución de problemas en los que interviene el concepto de Ecuación Diferencial Ordinaria: Un estudio exploratorio”, in Investigación en Educación Matemática. Comunicaciones de los grupos de investigación. XI Simposio de la SEIEM, M. Camacho, P. Bolea, P. Flores, B. Gómez, J. Murillo and Mª González, Eds. Tenerife, pp. 87-106, 2008.

[6] R. Davis, Learning Mathematics. The cognitive Science Approach to Mathematics Education, Croom Helm, Australia, 1984.

[7] C. Esteley and M. Villareal, “Análisis y categorización de errores en Matemática”, Revista de Educación Matemática, 11(1), pp. 16-35, 1996.

[8] C. Guerrero, M. Camacho and H.R. Mejía, “Dificultades de los estudiantes en la interpretación de las soluciones de ecuaciones diferenciales ordinarias que modelan un problema”, Enseñanza de las Ciencias, 28(3), pp. 341-352, 2010.

[9] S. Habre, “Exploring students ‘ strategies to solve ordinary differential equations in a reformed setting”, Journal of Mathematical Behavior, 18(4), pp. 455-472, 2000.

[10] A. Hernández, “Obstáculos en la Articulación de los Marcos Numérico, Gráfico y Algebraico en relación con las Ecuaciones Diferenciales”, unpublished.

[11] O. Kwon, C. Rasmussen and K. Allen, “Students’ Retention of Mathematical Knowledge and Skills in Differential Equations”, School Science and Mathematics, 105(5), pp. 1-13, 2005.

[12] L. Mammino, “Empleo del análisis de errores para aclarar conceptos de química general”, Enseñanza de las Ciencias, 20 (1), pp. 167-173, 2002.

[13] H. Radatz, “Student’s errors in the mathematical learning process: a survey”, *For the learning of Mathematics*, 1(1), pp. 16-20, 1980.

[14] C. Rasmussen, “New directions in differential equations: A framework for interpreting students ‘ understandings and difficulties”, Journal of Mathematical Behavior, 20(1), pp. 55-87, 2001.

[15] C. Rasmussen and H. Blumenfeld, “Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions”, Journal of Mathematical Behavior, 26, pp. 195-210, 2007.

[16] C. Rasmussen and O. Kwon, “An inquiry-oriented approach to undergraduate mathematics”, Journal of Mathematics Behavior, 26, pp. 189-194, 2007.

[17] C. Rasmussen, O. Kwon, K. Allen, K. Marrongelle and M. Burtch, “Capitalizing on Advances in Mathematics and K-12 Mathematics Education in Undergraduate Mathematics: An Inquiry-Oriented Approach to Differential Equations”, Asia Pacific Education Review, 7(1), pp. 85-93, 2006.

[18] C. Rasmussen and K. Whitehead, “Learning and Teaching Ordinary Differential Equations”, in MAA Online Research Sampler, A. Selden and J. Selden, Eds., 2003.

[19] L. Rico, “Errores y dificultades en el aprendizaje de las Matemáticas”, in Educación matemática, J. Kilpatrick, L. Rico and P. Gómez, Eds. México: Grupo Editorial Iberoamérica, pp. 69-108, 1998.

[20] M. Socas, "Dificultades, obstáculos y errores en el aprendizaje de las matemáticas en la Educación Secundaria", in La Educación Matemática en la Enseñanza Secundaria, L. Rico (Coord.). Barcelona: ICE Universitat de Barcelona/HORSORI, pp. 125-154, 1997.